NAG Toolbox for MATLAB

d05aa

1 Purpose

d05aa solves a linear, nonsingular Fredholm equation of the second kind with a split kernel.

2 Syntax

```
[f, c, ifail] = d05aa(lambda, a, b, k1, k2, g, n, ind)
```

3 Description

d05aa solves an integral equation of the form

$$f(x) - \lambda \int_a^b k(x,s) f(s)\, ds = g(x)$$

for $a \le x \le b$, when the kernel k is defined in two parts: $k = k_1$ for $a \le s \le x$ and $k = k_2$ for $x < s \le b$. The method used is that of El–Gendi 1969 for which, it is important to note, each of the functions k_1 and k_2 must be defined, smooth and nonsingular, for all x and s in the interval $[a, b]$.

An approximation to the solution $f(x)$ is found in the form of an n term Chebyshev-series $\sum_{i=1}^{n}{}' c_i T_i(x)$, where $'$ indicates that the first term is halved in the sum. The coefficients c_i, for $i = 1, 2, \ldots, n$, of this series are determined directly from approximate values f_i, for $i = 1, 2, \ldots, n$, of the function $f(x)$ at the first n of a set of $m + 1$ Chebyshev points:

$$x_i = \tfrac{1}{2}(a + b + (b - a)\cos[(i - 1)\pi/m]), \qquad i = 1, 2, \ldots, m + 1.$$

The values f_i are obtained by solving simultaneous linear algebraic equations formed by applying a quadrature formula (equivalent to the scheme of Clenshaw and Curtis 1960) to the integral equation at the above points.

In general $m = n - 1$. However, if the kernel k is centro-symmetric in the interval $[a, b]$, i.e., if $k(x, s) = k(a + b - x, a + b - s)$, then the function is designed to take advantage of this fact in the formation and solution of the algebraic equations. In this case, symmetry in the function $g(x)$ implies symmetry in the function $f(x)$. In particular, if $g(x)$ is even about the mid-point of the range of integration, then so also is $f(x)$, which may be approximated by an even Chebyshev-series with $m = 2n - 1$. Similarly, if $g(x)$ is odd about the mid-point then $f(x)$ may be approximated by an odd series with $m = 2n$.

4 References

Clenshaw C W and Curtis A R 1960 A method for numerical integration on an automatic computer *Numer. Math.* **2** 197–205

El–Gendi S E 1969 Chebyshev solution of differential, integral and integro-differential equations *Comput. J.* **12** 282–287

5 Parameters

5.1 Compulsory Input Parameters

1: **lambda – double scalar**

The value of the parameter λ of the integral equation.

2: **a – double scalar**

a, the lower limit of integration.

3: **b – double scalar**

b, the upper limit of integration.

Constraint: **b** $>$ **a**.

4: **k1 – string containing name of m-file**

k1 must evaluate the kernel $k(x,s) = k_1(x,s)$ of the integral equation for $a \leq s \leq x$.

Its specification is:

```
[result] = k1(x, s)
```

Input Parameters

1: **x – double scalar**
2: **s – double scalar**

The values of x and s at which $k_1(x,s)$ is to be evaluated.

Output Parameters

1: **result – double scalar**

The result of the function.

5: **k2 – string containing name of m-file**

k2 must evaluate the kernel $k(x,s) = k_2(x,s)$ of the integral equation for $x < s \leq b$.

Its specification is:

```
[result] = k2(x, s)
```

Input Parameters

1: **x – double scalar**
2: **s – double scalar**

The values of x and s at which $k_2(x,s)$ is to be evaluated.

Output Parameters

1: **result – double scalar**

The result of the function.

Note that the functions k_1 and k_2 must be defined, smooth and nonsingular for all x and s in the interval $[a,b]$.

6: **g – string containing name of m-file**

g must evaluate the function $g(x)$ for $a \leq x \leq b$.

Its specification is:

```
[result] = g(x)
```

Input Parameters

1: **x – double scalar**

The values of x at which $g(x)$ is to be evaluated.

Output Parameters

1: **result – double scalar**

The result of the function.

7: **n – int32 scalar**

the number of terms in the Chebyshev-series required to approximate $f(x)$.

Constraint: $\mathbf{n} > 0$.

8: **ind – int32 scalar**

Must be set to 0, 1, 2 or 3.

$\mathbf{ind} = 0$

$k(x, s)$ is not centro-symmetric (or no account is to be taken of centro-symmetry).

$\mathbf{ind} = 1$

$k(x, s)$ is centro-symmetric and $g(x)$ is odd.

$\mathbf{ind} = 2$

$k(x, s)$ is centro-symmetric and $g(x)$ is even.

$\mathbf{ind} = 3$

$k(x, s)$ is centro-symmetric but $g(x)$ is neither odd nor even.

5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

w1, w2, wd, ldw1, ldw2

5.4 Output Parameters

1: **f(n) – double array**

The approximate values f_i, for $i = 1, 2, \ldots, \mathbf{n}$ of $f(x)$ evaluated at the first **n** of $\mathrm{M} + 1$ Chebyshev points x_i, (see Section 3).

If $\mathbf{ind} = 0$ or 3, $\mathrm{M} = \mathbf{n} - 1$.

If $\mathbf{ind} = 1$, $\mathrm{M} = 2 \times \mathbf{n}$.

If $\mathbf{ind} = 2$, $\mathrm{M} = 2 \times \mathbf{n} - 1$.

2: **c(n) – double array**

The coefficients c_i, for $i = 1, 2, \ldots, \mathbf{n}$ of the Chebyshev-series approximation to $f(x)$.

If **ind** is 1 this series contains polynomials of odd order only and if **ind** is 2 the series contains even order polynomials only.

3: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, $\mathbf{a} \geq \mathbf{b}$ or $\mathbf{n} < 1$.

ifail $= 2$

A failure has occurred due to proximity to an eigenvalue. In general, if **lambda** is near an eigenvalue of the integral equation, the corresponding matrix will be nearly singular. In the special case, $m = 1$, the matrix reduces to a zero-valued number.

7 Accuracy

No explicit error estimate is provided by the function but it is usually possible to obtain a good indication of the accuracy of the solution either

(i) by examining the size of the later Chebyshev coefficients c_i, or

(ii) by comparing the coefficients c_i or the function values f_i for two or more values of **n**.

8 Further Comments

The time taken by d05aa increases with **n**.

This function may be used to solve an equation with a continuous kernel by defining user-supplied real function **k1** and user-supplied real function **k2** to be identical.

This function may also be used to solve a Volterra equation by defining user-supplied real function **k2** (or user-supplied real function **k1**) to be identically zero.

9 Example

d05aa_g.m

```
function [result] = g(x)
  result = sin(pi*x)*(1.0d0-1.0d0/(pi*pi));
```

d05aa_k1.m

```
function [result] = k1(x, s)
  result = s*(1.0d0-x);
```

d05aa_k2.m

```
function [result] = k2(x, s)
  result = x*(1.0d0-s);
```

```
lambda = 1;
a = 0;
b = 1;
n = int32(5);
```

```
ind = int32(2);
[f, c, ifail] = d05aa(lambda, a, b, 'd05aa_k1', 'd05aa_k2', 'd05aa_g', n,
ind)
```

```
f =
    0.0000
    0.0946
    0.3593
    0.7071
    0.9630
c =
    0.9440
   -0.4994
    0.0280
   -0.0006
    0.0000
ifail =
           0
```